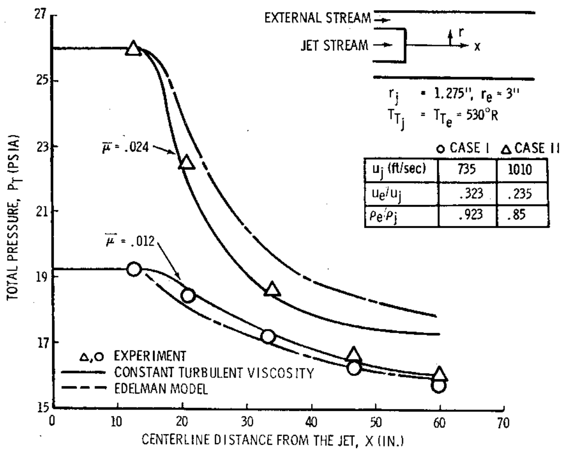

Fig. 1 Total pressure distributions along the centerline of the duct.

In the theoretical solution it is assumed that the turbulent viscosity is known a priori. In this study Ferri's[4] model is used in the potential core region

$$\bar{\mu} = \kappa \bar{x}(\overline{(\rho u)}_e - 1) \tag{1}$$

where κ is a constant.

In the main mixing region two different models are used. The first assumes that the turbulent viscosity is constant

$$\bar{\mu} = \text{const} \tag{2}$$

The second is the model employed by Edelman[5]:

$$\bar{\mu} = 0.018\bar{r}_{1/2}(\bar{\rho}\bar{u})_L \tag{3}$$

where $\bar{r}_{1/2}$ is the nondimensionalized half-radius defined by the location of the mean mass flow rate per unit area across the duct. It is also assumed that the turbulent Prandtl and Schmidt numbers are equal to 1.

Comparison of Theory and Experiment

Experimental centerline distributions of total pressure for air/air jet mixing obtained by John[6] in a constant area duct are compared with theoretical calculations in Fig. 1. The initial conditions in the jet and external stream are also presented in the figure. For both cases the constant in the potential core viscosity model was evaluated using a freejet mixing analysis. For the first case $\kappa = 0.0008$, and for the second $\kappa = 0.0005$. Also, the length of the potential core was taken from the experiment. For the first case ($u_j = 735$ fps) $\bar{\mu} = 0.012$ in the main mixing region results in a theoretical distribution which is in good agreement with the experimental distribution. Use of the Edelman model also results in good agreement between theory and experiment. In the second case ($u_j = 1010$ fps) $\bar{\mu} = 0.024$ gives good agreement between theory and experiment. For this latter case the Edelman model predicts too slow of a mixing process.

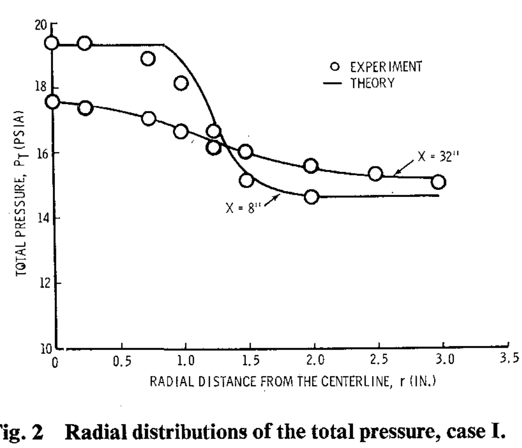

Fig. 2 Radial distributions of the total pressure, case I.

Experimental radial distributions of total pressure are compared with theoretical calculations in Fig. 2 for case I. The theory utilized Eqs (1) and (2). The agreement between theory and experiment is good for both the potential core and main mixing regions.

Certainly the good agreement between theory and experiment demonstrates the ability to predict coaxial jet mixing in a constant area duct. However, it is felt that extensive data analysis is needed to determine a reliable turbulent viscosity model for this type of mixing process.

References

[1] Schetz, J., "Analysis of the Mixing and Combustion of Gaseous and Particle-Laden Jets in an Airstream," AIAA Paper 69-33, New York, 1969.

[2] Harsha, T., "Free Turbulent Mixing: A Critical Evaluation of Theory and Experiment," AEDC-TR-71-36, Feb. 1971. Arnold Engineering Development Center, Arnold Air Force Base, Tenn.

[3] Baronti, P. and Rotta, N., "An Introduction to Mixing and Combustion Inside Channels," ATL-TR-126, Dec. 1968, Jericho, N.Y.

[4] Ferri, A., Libby, P., and Zakkay, V. "Theoretical and Experimental Investigation of Supersonic Combustion," Rept. 713, Sept. 1962, Polytechnic Inst. of Brooklyn, Brooklyn, N.Y.

[5] Edelman, R., and Fortune, O., "An Analysis of Mixing and Combustion in Ducted Flows," AIAA Paper 68-114, New York, 1968.

[6] John J., "An Experimental Investigation of the Bounded Mixing of Two Compressible Axially Symmetric Jet Streams," Aero Engineering Lab. Rept. 399, June 1957, Princeton Univ., Princeton, N.J.

Interpolation Using Surface Splines

ROBERT L. HARDER*
MacNeal-Schwendler Corp.
AND
ROBERT N. DESMARAIS†
NASA Langley Research Center

I. Introduction

A SURFACE spline is a mathematical tool for interpolating a function of two variables. It is based upon the small deflection equation of an infinite plate. The method was originally developed for interpolating wing deflections and computing slopes for aeroelastic calculations. The main advantages of the surface spline are that the coordinates of the known points need not be located in a rectangular array and the function may be differentiated to find slopes.

A linear spline, which is based upon the small deflection equation of an infinite beam, has been quite useful for one-dimensional interpolation. A lattice of linear splines has been used[1] to solve the two-dimensional problem. An advantage of the surface spline method it is that does not require the user to locate the splines.

Received October 1, 1971.
* Staff Scientist. Member AIAA.
† Aerospace Engineer, Loads Division.

The surface spline depends upon the solution of a system of linear equations, and thus, will ordinarily require the use of a digital computer. The closed form solution involves no functions more complicated than logarithms, and is easily coded.

II. Mathematical Analysis

The surface spline is a plate of infinite extent that deforms in bending only. The differential equation relating bending deflections and loads of a plate is

$$D\nabla^4 W = q \tag{1}$$

Deflections are specified at N independent points (x_i, y_i) $i = 1, N$. This requires point loads P_i at these N points. The values of these loads must be determined to give the specified deflections.

The first step is to find the deflection due to a point load. Introduce polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$), and determine the symmetric deflection due to a point load at the origin. Integrating Eq. (1),

$$W(r) = A + Br^2 + (P/16\pi D)r^2 \ln r^2 \tag{2}$$

In Eq. (2), A and B are undetermined coefficients, and P is the point load. The deflection of the entire spline will be taken as the sum of solutions of Eq. (2).

$$W(x,y) = \sum_{i=1}^{N} (A_i + B_i r_i + (P_i/16\pi D)r^2 \ln r^2) \tag{3}$$

where

$$r_i^2 = (x - x_i)^2 + (y - y_i)^2$$

The surface spline should become "flat" a long distance from the applied loads. Put $x = r\cos\theta$, $y = r\sin\theta$, and expand Eq. (3) for large r.

$$\begin{aligned} W(r, \theta) = {} & r^2 \ln r^2 \sum_{i=1}^{N} (P_i/16\pi D) + r^2 \sum_{i=1}^{N} B_i \\ & - 2r \ln r^2 \sum_{i=1}^{N} (x_i \cos\theta + y_i \sin\theta)(P_i/16\pi D) \\ & + 2r \sum_{i=1}^{N} (x_i \cos\theta + y_i \sin\theta)(P_i/16\pi D - B_i) \\ & + \ln r^2 \sum_{i=1}^{N} (x_i^2 + y_i^2)(P_i/16\pi D) + \cdots \end{aligned} \tag{4}$$

The remaining terms are of order $1, r^{-1}, r^{-2}$, etc. Terms of order $r^2 \ln r^2$, r^2, and $r \ln r^2$ can be eliminated by setting

$$\sum P_i = 0 \tag{5}$$

$$\sum x_i P_i = 0 \tag{6}$$

$$\sum y_i P_i = 0 \tag{7}$$

$$\sum B_i = 0 \tag{8}$$

Equations (5)–(7) are recognized as the equilibrium equations. Thus the actual condition for large r is that W contains terms of order $x = r\cos\theta$, $y = r\sin\theta$, $\ln r^2$, 1, etc.

The resulting equations are rearranged into a form useful for computation.

$$W(x,y) = a_0 + a_1 x + a_2 y + \sum_{i=1}^{N} F_i r_i^2 \ln r_i^2 \tag{9}$$

where $r_i^2 = (x - x_i)^2 + (y - y_i)^2$. To get from Eq. (3) to Eq. (9), let

$$a_0 = \sum_{i=1}^{N} [A_i + B_i(x_i^2 + y_i^2)], \quad a_2 = -2\sum_{i=1}^{N} B_i y_i,$$

$$a_1 = -2\sum_{i=1}^{N} B_i x_i, \quad F_i = P_i/16\pi D,$$

and use Eq. (8). The $N + 3$ unknowns are determined from

$$\sum_{i=1}^{N} F_i = \sum_{i=1}^{N} x_i F_i = \sum_{i=1}^{N} y_i F_i = 0 \tag{10}$$

and

$$W_j = a_0 + a_1 x_j + a_2 y_j + \sum_{i=1}^{N} F_i r_{ij}^2 \ln r_{ij}^2 \text{ for } (j = 1, N) \tag{11}$$

where $r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$. In order to use a surface spline, three or more noncolinear points (x_i, y_i) must be specified. Some care is required when applying the formula for $r = 0$, since

$$\lim_{r \to 0} r \ln r^2 = 0 \tag{12}$$

even though $\ln r^2$ does not exist. Equation (9) can be differentiated to give (for example)

$$\frac{\partial w}{\partial x}(x,y) = a_1 + 2\sum_{i=1}^{N} F_i(1 + \ln r_i^2)(x - x_i) \tag{13}$$

III. Results

Figures 1 and 2 show comparisons of surface-spline interpolations with interpolations by 21-term least-squares polynomials.† For each function, the surface was fitted at the

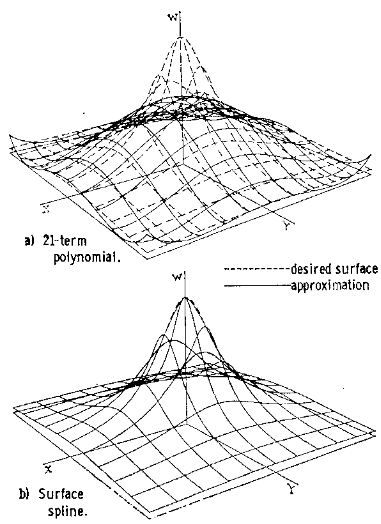

Fig. 1 Comparison of a 21-term least-squares polynomial approximation with surface spline approximation to a bell-shaped surface.

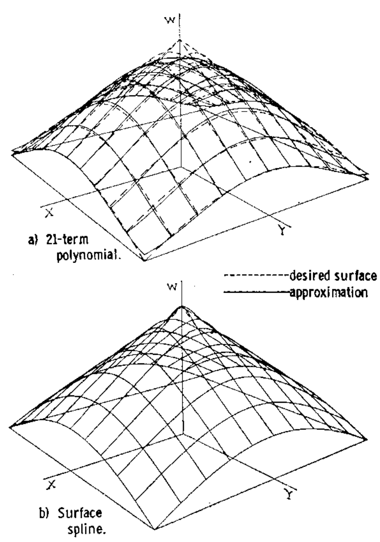

Fig. 2 Comparison of a 21-term least-squares polynomial approximation with surface spline approximation to part of a cone.

† The order of the polynomial was chosen as near optimum for good fit. Such polynomials have been used for interpolating deflections and calculating slopes of wings.

121 points located at the intersections of the curves. In Fig. 1, the bell-shaped surface

$$W(x,y) = (1 + 9x^2 + 16y^2)^{-1} \tag{14}$$

is difficult to approximate with a polynomial due to the Runge effect (i.e., it has poles nearby in the complex plane). Figure 2 is a part of the cone

$$W(x,y) = 1 - ((x^2 + y^2)/2)^{1/2} \tag{15}$$

The superiority of the surface spline is clearly indicated in both figures.

IV. Modifications

There are several modifications which can be incorporated.

a) Scaling. If the points (x_i, y_i) lie within a long narrow zone, a linear transformation can be and ordinarily should be made to transform the points into a nearly rectangular zone. This is illustrated in Fig. 3.

b) Symmetry. If one or two planes of symmetry or antisymmetry exist, then use can be made of images either to improve accuracy or to reduce the number of simultaneous equations. For example, if $W(x,y)$ is symmetric about $x = 0$, then replace Eq. (9) with

$$W(x,y) = a_0 + a_2 y + \sum_{i=1}^{N} F_i(r_i^2 \ln r_i^2 + \bar{r}_i^2 \ln \bar{r}_i^2) \tag{16}$$

where $\bar{r}_i^2 = (x + x_i)^2 + (y - y_i)^2$. Then set $\sum F_i = \sum y_i F_i = 0$, and $W_j = W(x_j, y_j)$ for the required $N + 2$ equations.

c) Smoothing with elastic springs. Instead of forcing the plate to pass through the N points, apply elastic spring forces to the plate that are proportional to the difference between the desired data point and the smoothed interpolated surface. Mathematically this is the same as Eq. (9) with

$$W_j = a_0 + a_1 x_j + a_2 y_j + \sum_{i=1}^{N} F_i r_{ij}^2 \ln r_{ij}^2 + C_j F_j \tag{17}$$

The coefficients C_j, which have units of length squared, are equal to $16\pi D/K_j$, where D is the plate rigidity and K_j is the spring constant associated with the jth point. For $C_j = 0$ $(K_j = \infty)$ we get the original spline, and for $C \to \infty$ $(K \to 0)$, the function (11) approaches a least-squares plane fit.

d) Smoothing with distributed loads. If the term $r^2 \ln r^2$ is replaced by $r^2 \ln(r^2 + \varepsilon)$ in both Eqs. (9) and (11), then a new surface is generated which passes through N points. If the loads are computed from Eq. (1), it follows that they are no longer point loads, but are distributed loads. The loads approach point loads as the parameter ε approaches zero. This procedure produces a surface for which all derivatives exist everywhere, and furthermore, it simplifies the coding somewhat.

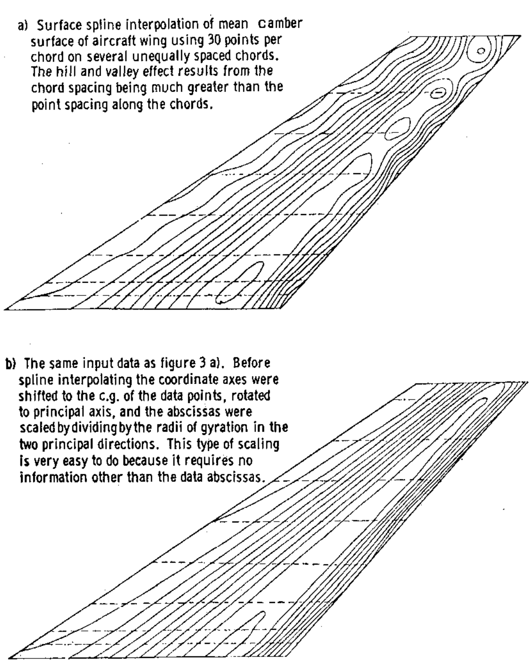

Fig. 3 Effect of scaling data abscissas.

References

[1] Done, G. T. S., "Interpolation of Mode Shapes: A Matrix Scheme Using Two-Way Spline Curves," *Aeronautical Quarterly*, Vol. XVI, Part 4, Nov. 1965, pp. 333–349.

Comment on "Transonic Airfoil Design"

J. Smolderen*
Von Kármán Institute for Fluid Dynamics,
Rhode Saint Genèse, Belgium

In their paper,[1] Cahn and Garcia present a method for the solution of the supercritical wing problem, based on a transformation of the polar hodograph coordinates, which reduces this problem to an incompressible potential flow calculation. Using this approach, a mixed elliptic-hyperbolic boundary problem, or even a purely hyperbolic problem could be reduced, in the general case, to a purely elliptic problem, and vice versa.

This result is clearly paradoxical as it contradicts well-known properties of elliptic and hyperbolic equations. For instance, using Cahn and Garcia's technique, it would become possible to construct well behaved solution of a Cauchy problem for an elliptic equation. One simply has to solve the problem in the supersonic region of the compressible hodograph plane, using, say, the method of characteristics, and then use the transformation (7) to obtain a reasonable solution of a Cauchy problem for the Laplace equation in the incompressible hodograph plane. If the Cauchy data used is bounded, but highly oscillatory in nature, then the solution of the hyperbolic problem in the compressible hodograph plane will also be bounded and the transformed solution will have the same property. However, the solution of a Cauchy problem with highly oscillating initial data for an elliptic equation cannot generally be bounded, because of the presence of highly divergent exponentials. This indicates that something of fundamental qualitative nature must be wrong in the method proposed by Cahn and Garcia.

In fact, it is obvious, by applying the transformation (7) to Eq. (2), that Eq. (1) are not recovered. The resulting system is elliptic as expected from the well-known property that the elliptic, hyperbolic, parabolic, or mixed character of a differential system is conserved by a transformation of independent variables. The transformation is thus meaningless and the paradox is resolved. The arguments proposed by Cahn and Garcia to justify the replacement of a well defined mixed type differential system by a completely different elliptic system are interesting and their fallacy can be shown only by subtler considerations.

Essentially, the algebra involved in deriving Eqs. (4–7) cover a basic assumption which is justified at the end of the section "Outline of Theory." The authors are aware of the

Received July, 7, 1971.

* Director; also, Professor of Applied Aerodynamics at the University of Liège, Belgium.